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Qualitative Capacities as Imprecise Possibilities

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Abstract. This paper studies the structure of qualitative capacities, that is, monotonic set-functions, when they range on a finite totally ordered scale equipped with an order-reversing map. These set-functions correspond to general representations of uncertainty, as well as importance levels of groups of criteria in multi-criteria decision-making. More specifically, we investigate the question whether these qualitative set-functions can be viewed as classes of simpler set-functions, typically possibility measures, paralleling the situation of quantitative capacities with respect to imprecise probability theory. We show that any capacity is characterized by a non-empty class of possibility measures having the structure of an upper semi-lattice. The lower bounds of this class are enough to reconstruct the capacity, and their number is characteristic of its complexity. We introduce a sequence of axioms generalizing the maxitivity property of possibility measures, and related to the number of possibility measures needed for this reconstruction. In the Boolean case, capacities are closely related to non-regular multi-source modal logics and their neighborhood semantics can be described in terms of qualitative Moebius transforms.

1 Introduction

A fuzzy measure (or a capacity) is a set-function that is monotonic under inclusion. If its range is a finite totally ordered scale, the capacity is said to be qualitative. Then, the connection with probability measures is lost as well, and a number of notions, meaningful in the quantitative setting, are lost, like the Möebius transform, the conjugate, nor can any qualitative capacity be viewed as encoding a family of probability distributions. Yet it seems that qualitative counterparts of many such quantitative notions can be defined if we replace probability measures by possibility measures. For instance the process of generation of belief functions, introduced by Dempster [6], was applied to possibility measures by Dubois and Prade [11,12] so as to define upper and lower possibilities and necessities. It was noticed that upper possibilities and lower necessities are still possibility and necessity measures respectively, but upper necessities and lower possibilities are not. This study was pursued by Tsiporkova and De Baets [21] in a more general setting. More recently in [18], it was shown that qualitative capacities can be viewed as counterparts of belief functions, using the possibilistic counterpart of a basic probability assignment. In [5] it was proved that the upper envelope of the possible extensions of a probability is a possibility measure.

A natural question is then whether a qualitative capacity can be viewed as a family of possibility measures as in Walley's theory of imprecise probability [22]. A recent paper [7] addressed this issue, taking up a pioneering work by Banon [2]. It is shown that in the case of qualitative information, special subsets of possibility measures play a role similar to convex sets of probability measures. This should not come as a surprise. Indeed, it has been shown that possibility measures can be refined by probability measures using a lexicographic refinement of the basic axiom of possibility measures, and that capacities on a finite set can be refined by belief functions [9,10]. The aim of this paper is to show that the maxitivity and minitivity axiom of possibility theory can be generalized to define families of qualitative capacities of increasing complexity. This property enables qualitative capacities to be seen as necessity modalities in a non-regular class of modal logics, extending the links between possibility theory and modal logic.

2 Capacities as Imprecise Possibilities and Necessities

Consider a finite set S and a finite totally ordered scale L with top 1 and bottom 0. A capacity (or fuzzy measure) is a mapping $\gamma : 2^S \to L$ such that $\gamma(\emptyset) = 0$; $\gamma(S) = 1$; and if $A \subseteq B$ then $\gamma(A) \leq \gamma(B)$. A special case of capacity is a possibility measure. In possibility theory, the available information is represented by means of a possibility distribution. This is a function, usually denoted π, from the universe of discourse S to the scale L. The function π is supposed to rank-order potential values of (some aspect of) the state of the world - according to their plausibility. The value $\pi(s)$ is understood as the possibility that s be the actual state of the world. Precise information corresponds to the situation where $\exists s^*, \pi(s^*) = 1$, and $\forall s \neq s^*, \pi(s) = 0$, while complete ignorance is represented by the vacuous possibility distribution $\pi^?$ such that $\forall s \in S, \pi^?(s) = 1$. The *possibility measure* is defined by $\Pi(A) = \max_{s \in A} \pi(s)$.

A possibility distribution π is said to be more specific than another possibility distribution ρ if $\forall s \in S, \pi(s) \leq \rho(s)$. Denote by γ^c the conjugate of γ, defined as $\gamma^c(A) = \nu(\gamma(A^c)), \forall A \subseteq S$, where A^c is the complement of set A, and ν the order-reversing map on L. The conjugate of a possibility measure is called a necessity measure. The conjugate necessity measure is then of the form $N(A) = \nu(\max_{s \notin A} \pi(s)) = \min_{s \notin A} N(S \setminus \{s\})$.

It is well-known that in the numerical setting some capacities g can be equivalently represented by a convex set of probabilities of the form $\mathcal{P}(g) = \{P, P(A) \geq g(A), \forall A \subseteq S\}$. For instance, g can be a convex capacity ($g(A \cup B) \geq g(A) + g(B) - g(A \cap B)$) or a belief function. Then it holds that $g(A) = \min\{P(A) : P \in \mathcal{P}(g)\}$. This is one example of a coherent lower probability in the sense of Walley [22] (exact capacity after Schmeidler [20]). In the qualitative case this construction is impossible. The natural question is then whether a similar construction may make sense with qualitative possibility measures replacing probability measures.

2.1 Imprecise Possibility and Necessity

There is always at least one possibility measure that dominates any capacity: the vacuous possibility measure, based on the distribution $\pi^?$ expressing ignorance, since then

$\forall A \neq \emptyset \subset S, \Pi(A) = 1 \geq \gamma(A), \forall$ capacity γ, and $\Pi(\emptyset) = \gamma(\emptyset) = 0$. Let

$$\mathcal{R}(\gamma) = \{\pi : \Pi(A) \geq \gamma(A), \forall A \subseteq S\}$$

be the set of possibility distributions whose corresponding set-functions Π dominate γ. We call $\mathcal{R}(\gamma)$ the *possibilistic credal set* induced by the capacity γ. In this section we recall some results on the structure of this set of possibility distributions.

Let σ be a permutation of the $n = |S|$ elements in S. The ith element of the permutation is denoted by $s_{\sigma(i)}$. Moreover let $S_\sigma^i = \{s_{\sigma(i)}, \ldots, s_{\sigma(n)}\}$. Define the possibility distribution π_σ^γ as follows:

$$\forall i = 1 \ldots, n, \pi_\sigma^\gamma(s_{\sigma(i)}) = \gamma(S_\sigma^i) \tag{1}$$

There are at most $n!$ (number of permutations) such possibility distributions. It can be checked that the possibility measure Π_σ^γ induced by π_σ^γ lies in $\mathcal{R}(\gamma)$ and that the $n!$ such possibility distributions enable γ to be reconstructed (already in [2]):

Proposition 1. *For each permutation* σ *:* $\forall A \subseteq S, \Pi_\sigma^\gamma(A) \geq \gamma(A)$.
Moreover, $\forall A \subseteq S, \gamma(A) = \min_\sigma \Pi_\sigma^\gamma(A)$

As a consequence,

Proposition 2. $\forall \pi \in \mathcal{R}(\gamma), \pi(s) \geq \pi_\sigma^\gamma(s), \forall s \in S$ *for some permutation* σ *of* S.

Proof: Just consider a permutation σ induced by π, that is $\sigma(i) \geq \sigma(j) \iff \pi(s_i) \leq \pi(s_j)$. For this permutation, $\Pi(S_\sigma^i) = \pi(s_i) \geq \gamma(S_\sigma^i) = \pi_\sigma^\gamma(s_i), \forall i = 1, \ldots, n$. ■

This result says that the possibility distributions π_σ^γ (we call the *marginals* of γ) include the least elements of $\mathcal{R}(\gamma)$ in the sense of fuzzy set inclusion, i.e., the most specific possibility distributions dominating γ. In other terms, $\mathcal{R}(\gamma) = \{\pi, \exists \sigma, \pi \geq \pi_\sigma^\gamma\}$. Of course the maximal element of $\mathcal{R}(\gamma)$ is the vacuous possibility distribution $\pi^?$. In the qualitative case, $\mathcal{R}(\gamma)$ is closed under the qualitative counterpart of a convex combination: if $\pi_1, \pi_2 \in \mathcal{R}(\gamma)$, then $\forall \alpha, \beta \in L$, such that $\max(\alpha, \beta) = 1$, it holds that $\max(\min(\alpha, \pi_1), (\min(\beta, \pi_2)) \in \mathcal{R}(\gamma)$. In fact $\mathcal{R}(\gamma)$ is an upper semi-lattice. Not all the $n!$ possibility distributions π_σ^γ are least elements of $\mathcal{R}(\gamma)$. As a trivial example, if $\gamma = \Pi$, this least element is unique and is precisely π. But other permutations yield other less specific possibility distributions.

Conversely, for any set $\mathcal{T}$ of possibility distributions, the set-function $\gamma(A) = \min_{\pi \in \mathcal{T}} \Pi(A)$ is a capacity. It is easy to see that $\mathcal{T} \subseteq \mathcal{R}(\gamma)$ and that if $\mathcal{T}$ contains only possibility distributions that are not comparable with respect to specificity, $\mathcal{T}$ forms the most specific elements of $\mathcal{R}(\gamma)$. Note that the set-function $\gamma(A) = \max_{\pi \in \mathcal{T}} \Pi(A)$ is not only a capacity, but also a possibility measure with possibility distribution $\pi^{\max}(s) = \max_{\pi \in \mathcal{T}} \pi(s)$ [13].

We denote by $\mathcal{R}_*(\gamma)$ the set of minimal elements in $\mathcal{R}(\gamma)$. They are by construction a finite set of possibility distributions none of which is more specific that another. It is clear that the complexity of a qualitative capacity is clearly measured by the number of elements in $\mathcal{R}_*(\gamma)$. These findings also show that any capacity can be viewed as a lower possibility measure:

$$\gamma(A) = \min\{\Pi(A), \pi \in \mathcal{R}_*(\gamma)\}.$$

This is similar to the case of a convex capacity g understood as a lower probability with respect to a (probabilistic) credal set $\mathcal{P}(g)$ [22]. This probability set forms a convex polyhedron whose vertices are among probability assignments P^{γ}_{σ} of the form $p^{\gamma}_{\sigma}(s_{\sigma(i)}) = g(S^i_{\sigma}) - g(S^{i+1}_{\sigma})$, and $\mathcal{P}(g)$ is the convex hull of these probabilities.

Dually, though, we can describe capacities as upper necessities by means of a family of necessity functions that stem from the lower possibility description of their conjugates. Then we can define two sets of possibility functions from γ:

– The set $\mathcal{R}(\gamma)$ of possibility measures that dominate γ;
– The set $\mathcal{R}(\gamma^c)$ of possibility measures that dominate its conjugate γ^c.

Clearly, possibility measures that dominate γ^c are conjugates of necessity measures dominated by γ. In other words γ is also an upper necessity measure in the sense that

$$\gamma(A) = \max\{N(A), \pi \in \mathcal{R}_*(\gamma^c)\}.$$

We can denote the set of minimal possibility distributions generating maximal necessity measures dominated by γ by $\mathcal{R}^*(\gamma) = \mathcal{R}_*(\gamma^c)$. One representation of γ (by means of $\mathcal{R}_*(\gamma)$ or $\mathcal{R}_*(\gamma^c)$) may be simpler than the other. For instance, if γ is a necessity measure based on possibility distribution π, then $\mathcal{R}^*(\gamma) = \{\pi\}$ while $\mathcal{R}_*(\gamma)$ contains several possibility distributions including π. Note that $\Pi(A) \geq N(A) = \Pi^c(A)$, so that it looks more natural to reach N from below and Π from above. More generally if a capacity γ is such that $\gamma(A) \geq \gamma^c(A), \forall A \subseteq S$, ($\gamma$ is an upper capacity) then it is clear that $\mathcal{R}_*(\gamma)$ is more natural than $\mathcal{R}_*(\gamma^c)$ for representing γ by a family of possibility measures that dominate it.

2.2 Generalized Minitivity and Maxitivity Axioms

For each capacity γ, there is a least integer n along with n necessity measures such that $\gamma(A) = \max_{i=1}^n N_i(A)$. We now show that this property can be described by means of an axiom of the form:

n-**adjunction**: $\forall A_i, i = 1, \ldots n+1, \min_{i=1}^{n+1} \gamma(A_i) \leq \max_{1\leq i<j\leq n+1} \gamma(A_i \cap A_j)$

that generalizes the minitivity axiom of necessity measures. Indeed, When $n = 1$, this is the usual adjunction property $\min(\gamma(A), \gamma(B)) \leq \gamma(A \cap B)$. It is then equivalent to the minitivity axiom of necessity measures: $N(A \cap B) = \min(N(A), N(B))$ since γ is inclusion-monotonic: 1-adjunctive capacities are necessity measures. Let us consider the next step: 2-adjunction.

Proposition 3. $\min(\gamma(A), \gamma(B), \gamma(C)) \leq \max(\gamma(A\cap B), \gamma(B\cap C), \gamma(A\cap C)), \forall A, B, C,$ *if and only if there exist two necessity measures such that* $\forall A, \gamma(A) = \max(N_1(A), N_2(A))$.

Proof:

$\Leftarrow$: Suppose $\gamma(A) = \max(N_1(A), N_2(A))$. We can assume without loss of generality that $N_1(A) \geq N_2(A), N_1(B) \geq N_2(B), N_2(C) \geq N_1(C)$ with one strict inequality, for some A, B, C (otherwise γ is a necessity measure) and then

$$\min(\gamma(A), \gamma(B), \gamma(C)) = \min(N_1(A), N_1(B), N_2(C))$$

follows. Now consider $\gamma(A \cap B)$. We have

$$\gamma(A \cap B) = \max(\min(N_1(A), N_1(B)), \min(N_2(A), N_2(B)).$$

Developing: $\gamma(A \cap B) = \min(\max(N_1(A), N_2(A)), \max(N_1(A), N_2(B)), \max(N_1(B), N_2(A)), \max(N_1(B), N_2(B))).$
Now since by construction $N_1(A) \geq N_2(A), N_1(B) \geq N_2(B)$, it follows that

$$\gamma(A \cap B) = \min(N_1(A), \max(N_1(A), N_2(B)), \max(N_1(B), N_2(A)), N_1(B))$$

$$= \min(N_1(A), N_1(B)) = \min(\gamma(A), \gamma(B)) \geq \min(\gamma(A), \gamma(B), \gamma(C))$$

Hence $\max(\gamma(A \cap B), \gamma(B \cap C), \gamma(B \cap C)) \geq \min(\gamma(A), \gamma(B), \gamma(C))$.

$\Rightarrow$: To get the converse, suppose that non trivially, $\gamma(A) = \max_{i=1}^{3} N_i(A)$. Then one may find distinct sets A, B, C such that

$$\min(\gamma(A), \gamma(B), \gamma(C)) = \min(N_1(A), N_2(B), N_3(C)).$$

It is easy to find an example for which $\min(\gamma(A), \gamma(B), \gamma(C)) > \max(\gamma(A \cap B), \gamma(B \cap C), \gamma(B \cap C))$. For instance,we can choose the three distinct sets A, B, C such that $\gamma(A) = N_1(A)$ and $\gamma(A') = 0, \forall A' \subset A$, $\gamma(B) = N_2(B)$ and $\gamma(B') = 0, \forall B' \subset B$, $\gamma(C) = N_3(C)$ and $\gamma(C') = 0, \forall C' \subset C$. These are the least elements of the family : $\{D, \gamma(D) > 0\}$ that forms a union of three filters exactly (they are the cores of the possibility distributions inducing necessity functions $N_i, i = 1, 2, 3$). It is then clear that A, B, C are not included into one another, so that $\max(\gamma(A \cap B), \gamma(B \cap C), \gamma(A \cap C)) = 0$. Indeed, for instance $A \cap B \subset A$ and $A \cap B \subset B$ (strict inclusion), and $\gamma(A \cap B) = 0$ by construction. The same reasoning holds for $B \cap C, A \cap C$. ■

Note that in general, if $\gamma(A) = \max(N_1(A), N_2(A))$, there can be a strict inequality $\min(\gamma(A), \gamma(B), \gamma(C)) < \max(\gamma(A \cap B), \gamma(B \cap C), \gamma(B \cap C))$. Indeed it is enough that $\gamma(C) < \gamma(A \cap B)$. It contrasts with the case of $n = 1$ that comes down to $\gamma(A \cap B) \geq \min(\gamma(A), \gamma(B))$ and implies $\gamma(A \cap B) = \min(\gamma(A), \gamma(B))$, due to monotonicity of γ.

In the general case, it holds that

Proposition 4. $\forall A_i, i = 1, \ldots n+1, \min_{i=1}^{n+1} \gamma(A_i) \leq \max_{i \neq j} \gamma(A_i \cap A_j)$ *if and only if there exist* n *necessity measures such that* $\forall A, \gamma(A) = \max_{j=1}^{n} N_j(A)$.

Proof

$\Leftarrow$: Suppose $\forall A, \gamma(A) = \max_{j=1}^{n} N_j(A)$. As a consequence:

$$\min_{i=1}^{n+1} \gamma(A_i) = \min_{i=1}^{n+1} \max_{j=1}^{n} N_j(A_i) = \min_{i=1}^{n+1} N_{j_i}(A_i)$$

where $N_{j_i}(A_i) \geq N_k(A_i), \forall k \neq j_i, k = 1, \ldots n, i = 1, \ldots n+1$. It is clear that at least two among indices $j_i, i = 1, n+1$ are equal, since there are only n distinct values of j. Suppose they are $j_1 = 1 = j_2$ without loss of generality, that is, $\min_{i=1}^{n+1} \gamma(A_i) = \min(N_1(A_1), N_1(A_2), \min_{i=3}^{n+1} N_{j_i}(A_i))$.

Now $\gamma(A_1 \cap A_2) = \max_{i=1}^n N_i(A_1 \cap A_2) = \max_{i=1}^n \min(N_i(A_1), N_i(A_2))$. However by assumption $N_1(A_1) \geq N_k(A_1), k = 2, \ldots n$ and $N_1(A_2) \geq N_k(A_2), k = 2, \ldots n$, so $\min(N_1(A_1), N_1(A_2)) \geq \min(N_k(A_1), N_k(A_2)), k = 2, \ldots n$. As a consequence, $\gamma(A_1 \cap A_2) = \min(N_1(A_1), N_1(A_2)) = \min(\gamma(A_1), \gamma(A_2)) \geq \min_{i=1}^{n+1} \gamma(A_i)$.

$\Rightarrow$: For the converse, the proof is the same as for the case $n = 3$: suppose that non trivially, $\gamma(A) = \max_{i=1}^{n+1} N_i(A)$. Then one may find a family of $n+1$ distinct sets A_i such that $\gamma(A_i) = N_i(A_i), i = 1, \ldots, n+1$ and also choose them such that

$$\min_{i=1}^{n+1} \gamma(A_i) > \max_{1 \leq i < j \leq n+1} \gamma(A_i \cap A_j).$$

Indeed, choose the $n+1$ distinct sets A_i with $\gamma(A_i) = N_i(A_i)$ and $\gamma(A) = 0, \forall A \subset A_i, i = 1, \ldots, n+1$. These are the least elements of the family: $\{D, \gamma(D) > 0\}$ that is formed by a union of $n+1$ filters exactly (they are the cores of the possibility distributions inducing $N_i, i = 1, n+1$). It is then clear that none of the $A_i's$ are included into one another, so that $\forall i < j, A_i \cap A_j \subset A_i$ and $A_i \cap A_j \subset A_j$ (strict inclusion) hence $\gamma(A_i \cap A_j) = 0$ by construction; so, $\max_{1 \leq i < j \leq n+1} \gamma(A_i \cap A_j) = 0$. ■

Note that if a capacity possesses n-adjunction it provides an upper bound on the number of its focal sets having a given weight. Indeed, if γ_λ denotes the Boolean capacity obtained as $\gamma_\lambda(A) = 1$ if $\gamma(A) \geq \lambda$, and 0 otherwise, then since $\gamma(A) = \max_{i=1}^n N_i(A)$, it follows that the set of focal sets of γ_λ is made of the n subsets E_i such that $N_i(A) \geq \lambda \iff E_i \subseteq A$.

In fact, if E is a focal set of γ, i.e. $E \in \mathcal{F}^\gamma$, define the necessity measure N_E by $\forall A \neq S, N_E(A) = \gamma_\#(E)$ if $E \subseteq A$ and 0 otherwise. It is clear that $\gamma(A) = \max_{E \in \mathcal{F}^\gamma} N_E(A)$. This is not the minimal form of course. To get the minimal form one may consider all chains of nested subsets in $\mathcal{F}^\gamma$: each such chain i defines a necessity measure N_i whose nested focal sets form the chain. If a capacity possesses n-adjunction, it means that there are exactly n chains of focal sets in $\mathcal{F}^\gamma$.

Note that in the extreme case where the focal sets in $\mathcal{F}^\gamma$ are singletons, each necessity measure N_E is also a possibility measure (it is a Dirac measure based on $E = \{s_E\}$), hence γ is a possibility measure.

Of course the above results can be adapted, replacing necessity measures by possibility measures, thus weakening the notion of maxitivity. We can consider the following axiom, dual to n-adjunction:

n-**max-dominance**: $\max_{i=1}^{n+1} \gamma(A_i) \geq \min_{1 \leq i < j \leq n+1} \gamma(A_i \cup A_j)$

$\forall A_i, i = 1, \ldots n+1$, and prove the counterpart to the above proposition:

Proposition 5. $\max_{i=1}^{n+1} \gamma(A_i) \geq \min_{i \neq j} \gamma(A_i \cup A_j)$ *if and only if if there exist* n *possibility measures such that* $\gamma(A) = \min_{i=1}^n \Pi_i(A)$.

Comment: In the numerical setting, the n-superadditivity of a capacity is implied by but does not imply its $(n+1)$-superadditivity. The above concept of n-minitivity (in fact n-adjunction) seems to play a similar role: we can generalize necessity functions by steps since n-minitivity implies, but is not implied by $(n+1)$-minitivity.

2.3 Qualitative Focal Sets, n-Adjunction and k-Maxitivity

The inner (qualitative) Moebius transform of a capacity γ is a mapping $\gamma_{\#} : 2^S \to L$ defined by

$$\gamma_{\#}(E) = \gamma(E) \text{ if } \gamma(E) > \max_{B \subsetneq E} \gamma(B) \tag{2}$$

and 0 otherwise. In the above definition, due to the monotonicity property, the condition $\gamma(E) > \max_{B \subsetneq E} \gamma(B)$ can be replaced by $\max_{x \in E} \gamma(E \setminus \{x\})$. It is easy to check that

- $\gamma_{\#}(\emptyset) = 0$; $\max_{A \subseteq S} \gamma_{\#}(A) = 1$;
- If $A \subset B$, and $\gamma_{\#}(A) > 0, \gamma_{\#}(B) > 0$, then $\gamma_{\#}(A) < \gamma_{\#}(B)$.

Let $\mathcal{F}^{\gamma} = \{E, \gamma_{\#}(E) > 0\}$ be the family of focal sets associated to γ. The last property says that the inner qualitative Moebius transform of γ is strictly monotonic with inclusion on $\mathcal{F}^{\gamma}$. It is clear that the inner qualitative Moebius transform of a possibility measure coincides with its possibility distribution: $\Pi_{\#}(A) = \pi(s)$ if $A = \{s\}$ and 0 otherwise. This property makes it clear that $\gamma_{\#}$ generalizes the notion of possibility distribution to the power set of S.

The inner (qualitative) Moebius transform contains the minimal information needed to reconstruct the capacity γ since, by construction [14,9]:

$$\gamma(A) = \max_{E \subseteq A} \gamma_{\#}(E) \tag{3}$$

The reader can check that if one of the values $\gamma_{\#}(E)$ is changed, the corresponding capacity will be different, namely the values $\gamma(A)$ such that $\gamma(A) = \gamma_{\#}(E)$. In a previous paper [7], it was shown that the qualitative Moebius transform is instrumental in finding the most specific possibility distributions dominating γ, via a selection process picking an element in each focal set.

The similarity between capacities and belief functions [19] is striking on the above equation: $\max$ replaces the sum in the expression of a belief function, and $\gamma_{\#}$ plays the role of the mass assignment, which is the Moebius transform of the belief function [15]. The subsets E in $\mathcal{F}^{\gamma}$ receive positive support and play the same role as the focal sets in Dempster-Shafer's theory: they are the primitive items of knowledge.

A capacity is said to be k-maxitive if and only if its focal sets have at most k elements. This notion was introduced by Mesiar [17] and Grabisch [14] as a class of simpler capacities. We show here a connection between the k-adjunction of capacities and the notion of k-maxitivity. The minitivity (1-adjunction) of necessity measures N go along with the fact that the focal elements of the conjugate possibility measure $\Pi(A) = \nu(N(A^c))$ are obviously the singletons $\{s\}$ such that $s \in A$ (1-maxitivity).

This construction can be generalized first to any qualitative capacity γ that ranges on $\{0, 1\}$. Let $\mathcal{F}^{\gamma}$ be its focal sets ($\gamma_{\#}(E) = 1$), and γ^c is its conjugate. Then obviously,

$$\gamma(A) = 1 \iff \exists E \in \mathcal{F}^{\gamma}, E \subset A \tag{4}$$

Lemma 1. *Suppose $\mathcal{F}^{\gamma} = \{E_1, \ldots E_k\}$ for a Boolean capacity γ. Then $\gamma^c(A) = 1$ if only if A contains a set the form $\{s_1, \ldots s_k\}, s_i \in E_i, i = 1 \ldots, k$.*

Proof: Indeed: $\gamma^c(A) = 1 \iff \gamma(A^c) = 0 \iff \forall E \in \mathcal{F}^\gamma, E \not\subseteq A^c$
hence: $\gamma^c(A) = 1 \iff \forall E \in \mathcal{F}^\gamma, E \cap A \neq \emptyset$. We can write this as follows:
$\gamma^c(A) = 1 \iff \forall E \in \mathcal{F}^\gamma, \exists s_E \in E \cap A \iff \exists F = \{s_E : E \in \mathcal{F}^\gamma\}, F \subseteq A$,
where for each focal set E of γ, s_E is picked in E. ■

Proposition 6. *The set of focal sets of γ^c is $\mathcal{F}^{\gamma^c} = \min_\subseteq\{\{s_1, \ldots s_k\}, s_i \in E_i, i = 1 \ldots, k\}$, where $\min_\subseteq$ picks the smallest subsets for inclusion.*

Proof: Note that $\mathcal{F}^{\gamma^c} = \min_\subseteq\{A, \gamma^c(A) = 1\}$. The result follows from Lemma 1. ■

Clearly, the elements s_E picked in focal sets E need not be distinct, in case the focal sets overlap. For instance, if $\mathcal{F}^\gamma = \{E_1, E_2\}$ with $E_1 = \{s_0, s_1, s_3\}, E_2 = \{s_0, s_2, s_4\}$, then the focal elements of the conjugate are the least elements among the family $\{\{s_0\}\} \cup \{\{s_0, s_i\}, i = 1, \ldots, 4\} \cup \{\{s_1, s_2\}, \{s_1, s_4\}, \{s_3, s_2\}, \{s_3, s_4\}\}$, that is $\mathcal{F}^{\gamma^c} = \{\{s_0\}\{s_1, s_2\}, \{s_1, s_4\}, \{s_3, s_2\}, \{s_3, s_4\}\}$.

Denoting by $c(\mathcal{F}^\gamma)$ the transformation from $\mathcal{F}^\gamma$ to $\mathcal{F}^{\gamma^c}$, we can prove:

Proposition 7. $c(c(\mathcal{F}^\gamma)) = \mathcal{F}^\gamma$

Proof: It is obvious because $(\gamma^c)^c = \gamma$. A direct proof is far less obvious.

For instance, if $\mathcal{F}^\gamma = \{A, B\}$. Then $\mathcal{F}^{\gamma^c} = \{\{s\} : s \in A \cap B\} \cup \{\{s_A, s_B\} : s_A \in A \setminus B, s_B \in B \setminus A\}$. To build dual focal sets from the latter family, each such focal set must contain $A \cap B$. Then suppose we pick $s_A \in \{s_A, s_B\}$. Clearly, this choice covers all focal sets $\{s_A, s\}, s \in B \setminus A$. It thus prevents us from picking the next element in $B \setminus A$. So the next elements to be picked lie in A. In fact, the focal sets left $\{s, s_B\}, s \neq s_A$ can be deprived of s_B since there is a focal set of the form $\{s_A, s_B\}$ that forbids s_B from further consideration. So this process reconstructs the focal set A.

From Prop. 6, it is clear that if a Boolean capacity is k-adjunctive (it has k focal sets), then its conjugate is k-maxitive, since the focal sets of its conjugate will have not more than k elements. In the next section, we shall see that the computation of the focal sets of a capacity from the ones of its conjugate corresponds in the modal logic setting to the swapping of modalities. In the following, we denote by $\mathcal{F}^\gamma_\beta$ the set of the focal elements A of a capacity γ such that $\gamma(A) = \beta$

Proposition 8. *For a general capacity γ, suppose $\mathcal{F}^\gamma = \{E_1, \ldots E_k\}$. Then, $\gamma^c(A) = 1$ if and only if $\forall i = 1 \cdots, k : E_i \cap A \neq \emptyset$. Moreover, $\mathcal{F}^{\gamma^c}_1 = \min_\subseteq\{\{s_1, \ldots s_k\}, s_i \in E_i, i = 1 \ldots, k\}$.*

Proof: It is like the proof of Lemma 1 and the subsequent proposition.

Lemma 2. *$\gamma^c(A) = \nu(\alpha) \neq 0, 1$ if and only if $\forall E, \gamma_\#(E) > \alpha$ implies $E \cap A \neq \emptyset$ and $\exists E, E \cap A = \emptyset$ such that $\gamma_\#(E) = \alpha$.*

Proof: $\gamma^c(A) \geq \nu(\alpha)$ if and only if $\gamma(A^c) \leq \alpha$ if and only if $\forall E, \gamma_\#(E) > \alpha$ implies $E \not\subseteq A^c$. Besides, the equality $\gamma^c(A) = \nu(\alpha)$ is attained if moreover there is a focal set $E \subseteq A^c$ such that $\gamma_\#(E) = \alpha$.

Proposition 9. *A is a focal element of γ^c such that $\gamma^c_{\#}(A) = \nu(\alpha) > 0$ if and only if it is a minimal element of the family $\{E = \{s_E : \gamma_{\#}(E) > \alpha\}, E \cap F = \emptyset$ for some $F \in \mathcal{F}^\gamma_\alpha\}$, where $s_E \in E$.*

Proof: A direct consequence of the lemma, since by construction $\gamma^c_{\#}(A) = \nu(\alpha)$ means that A is a minimal set such that $\gamma^c(A) = \nu(\alpha)$.

These results show how the inner qualitative Moebius transform of a capacity can be computed from the one of its conjugate. It is easy to see that also in the general case, if a capacity has k weighted focal sets, its conjugate will be k-maxitive, since the largest focal elements of γ^c (they have weight equal to 1) are obtained by picking one element in each focal set of γ. Another issue is now to compute the n possibility distributions such that γ is n-adjunctive in terms of the m possibility distributions such that γ is m-max-dominant. For instance, while a necessity measure N is 1-adjunctive w.r.t. its associated possibility distribution π, it is also n-max-dominant with respect to n possibility measures, where n is the number of (nested) focal sets of the necessity measure N. They are all distinct sets $A_{\alpha_i} = \{s : \pi(s) \geq \alpha_i\}$ such that $N_{\#}(A_{\alpha_i}) = \nu(\alpha_{i+1})$, where $\alpha_1 = 1 > \alpha_2 > \cdots > \alpha_n > \alpha_{n+1} = 0$. Then $N = \min_{i=1}^n \Pi_i$, where $\pi_i(s) = \nu(\alpha_{i+1}), \forall s \in A_{\alpha_i}$ and 1 otherwise.

3 The Modal Logic View of Capacities

In this section, we show that our previous results suggest a new semantics for general modal logics. Consider a propositional language $\mathcal{L}$ with Boolean variables $\{a, b, c...\}$ and standard connectives $\wedge, \vee, \neg, \rightarrow$. Let S be the set of interpretations of this language (assigning 1 or 0 to all variables). Given a proposition $p \in \mathcal{L}$, necessity measure N on S based on possibility distribution π, we denote by $\Box p$ the statement $N(A) \geq \lambda > 0$, where $A = [p]$ is the set of models of p. $\Box p$ corresponds to a Boolean necessity measure based on a possibility distribution that is the characteristic function of $E = \{s | \pi(s) > \nu(\lambda)\}$. Consider a higher level propositional language $\mathcal{L}_\Box$ defined by: $\forall p \in \mathcal{L}, \Box p \in \mathcal{L}_\Box$, and if $\phi, \psi \in \mathcal{L}_\Box$, then $\neg\phi \in \mathcal{L}_\Box$, and $\phi \wedge \psi \in \mathcal{L}_\Box$. The variables of $\mathcal{L}_\Box$ are thus $\{\Box p : p \in \mathcal{L}\}$. Let $\Diamond p$ be short for $\neg\Box\neg p$. Then $\models \Diamond p$ stands for $\Pi(A) \geq \nu(\lambda)$ where Π is the conjugate of N. It defines a very elementary fragment of a KD modal logic known as MEL [1]. Indeed, the following KD axioms are valid

- $(K) : \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
- $(N) : \ \Box\top$
- $(D) : \ \Box p \rightarrow \Diamond p$

and imply axiom (C) : $\Box(p \wedge q) \equiv (\Box p \wedge \Box q)$, which is the Boolean form of the minitivity axiom.

A "model" of a formula in $\phi \in \mathcal{L}_\Box$ is a nonempty subset $E \subseteq S$ of propositional models. The set E is understood as an epistemic state (a *meta-model*). The satisfaction of MEL-formulae is then defined recursively given $\phi, \psi \in \mathcal{L}_\Box$:

- $E \models \Box p$, if and only if $E \subseteq [p]$
- $E \models \neg\phi$, if and only if $E \not\models \phi$,
- $E \models \phi \wedge \psi$, if and only if $E \models \phi$ and $E \models \psi$,
- So, $E \models \Diamond p$ if and only if $E \cap [p] \neq \emptyset$

For any set $\Gamma \cup \{\phi\}$ of $\mathcal{L}_\Box$-formulae, ϕ is a semantic consequence of Γ, written $\Gamma \models \phi$, provided for every epistemic state E, $E \models \Gamma$ implies $E \models \phi$. This Boolean possibilistic logic, equipped with modus ponens, (the $\mathcal{L}_\Box$-fragment of KD) is sound and complete w.r.t. this semantics [1]. In fact, if N is the Boolean necessity measure induced by E, it defines precisely a classical interpretation of $\mathcal{L}_\Box$, of the form $\bigwedge_{p\in\mathcal{L}:N([p])=1} \Box p \wedge \bigwedge_{p\in\mathcal{L}:N([p])=0} \neg\Box p$ obeying axioms K, D, N. In particular the semantics does not rely on the use of accessibility relations.

Using the same language, denote now $\models \Box p$ as standing for $\gamma([p]) \geq \lambda > 0$ for any qualitative capacity γ. $\Box p$ now corresponds to a Boolean capacity defined by $\gamma_\lambda(A) = 1$ if $\gamma([p]) \geq \lambda > 0$ and 0 otherwise. The following axioms are then verified [8]:

- $(RE):\ \Box p \equiv \Box q$ whenever $\vdash p \equiv q$.
- $(RM):\ \Box p \rightarrow \Box q$, whenever $\vdash p \rightarrow q$.
- $(N):\ \ \Box\top;\ \ (P):\ \ \Diamond\top$.

It is a non-regular modal logic. It is a fragment of the *monotonic modal* logic EMN, Chellas [4], where modalities only apply to propositions. Its usual semantics is based on so-called neighborhoods (families of subsets of possible worlds having some properties). This logic no longer satisfies axioms K, C nor D. This modal logic is the natural logical account of qualitative capacities. Indeed, any classical interpretation of $\mathcal{L}_\Box$ that satisfies the above axioms defines and is defined by a Boolean capacity β and is of the form $\bigwedge_{p\in\mathcal{L}:\beta([p])=1} \Box p \wedge \bigwedge_{p\in\mathcal{L}:\beta([p])=0} \neg\Box p$.

Interestingly, we can capture the n-adjunction axiom in the modal setting (see [8] for $n = 2$). Let n be the smallest integer for which $\gamma(A) = \max_{i=1}^n N_i(A)$. Denoting by $\Box_i p$ the statement $N_i([p]) \geq \lambda > 0$, it is clear that $\gamma([p]) \geq \lambda > 0$ stands for $\Box p \equiv \vee_{i=1}^n \Box_i p$, where $\Box_i$ are KD modalities. By duality we can define $\Diamond p$ as short for $\neg\Box\neg p$, that is, $\Diamond p \equiv \wedge_{i=1}^n \Diamond_i p$. So, applying the characterisation of n-minitivity to the restriction of the modal logic EMN yields the axiom

$$(n\text{-C}):\ \vdash (\wedge_{i=1}^{n+1} \Box p_i) \rightarrow \vee_{i\neq j=1}^{n+1} \Box(p_i \wedge p_j)$$

It implies that if $p_i, i = 1\ldots, n+1$ are mutually inconsistent, then $\vdash \neg \wedge_{i=1}^{n+1} \Box p_i$. This property claims that we cannot have $\gamma([p_i]) \geq \lambda > 0$ for all $i = 1\ldots, n+1$.

The semantics of the EMNP$+n$-C logic can be expressed in two ways:

- In terms of n-tuple of epistemic states (subsets of S) : $(E_1, \ldots, E_n) \models \Box p$ if $\exists i \in [1, n], E_i \models \Box_i p$. By construction, $E_1, \ldots, E_n$ are the focal sets of the Boolean capacity defined by $\gamma_\lambda(A) = 1$ if $\gamma([p]) \geq \lambda > 0$ and 0 otherwise.
- More classically, in terms of neigborhoods: they are non-empty subsets $\mathcal{N}$ of 2^S such that $\mathcal{N} \models \Box p$ if and only if $[p] \in \mathcal{N}$ and $\mathcal{N} \models \Diamond p$ if and only if $[\neg p] \notin \mathcal{N}$.

For a KD modality, it is obvious that $\mathcal{N} = \{A, N(A) \geq \lambda\} = \{A | A \supseteq E\}$ for some non-empty $E \subseteq S$ ($\mathcal{N}$ is a proper filter). For an EMNP modality $\mathcal{N} = \{A, \gamma(A) \geq \lambda > 0\} \neq 2^S$ is closed under inclusion and not empty). For an EMNP$+n$-C modality, $\mathcal{N} = \{A, \gamma(A) \geq \lambda > 0\}$ is the union of n proper filters of the form $\{A, N_i(A) \geq \lambda\} = \{A | A \supseteq E_i\}$.

In the extreme case when the sets $(E_1, \ldots, E_n)$ are singletons (i.e., fully informed conflicting sources), the necessity modality $\Box p$ satisfies distributivity w.r.t. disjunction: $\vdash \Box(p \vee q) \equiv \Box p \vee \Box q$ (but no longer w.r.t. conjunction !) and the opposite of axiom D : $\vdash \Diamond p \rightarrow \Box p$. In other words, necessity and possibility modalities are exchanged.

We go back to the MEL logic exchanging the basic modalities $\Box$ and $\Diamond$. In fact, the swapping of modalities is a simple instance of the more general question, considered in the previous section, of computing the focal sets of a capacity from the ones of its conjugate. It comes down at the semantic level to the transformation of a logic based on the epistemic states of k agents into the dual situation of multiple source epistemic logic underlying a set of agents whose knowledge has limited imprecision (i.e., each epistemic state involves at most k possible worlds).

4 Conclusion

We have studied the representation of capacities having values on a finite totally ordered scale by families of qualitative possibility distributions. It turns out that any capacity can be viewed either as a lower possibility measure or as an upper necessity measure with respect to two distinct families of possibility distributions. This remark has led to propose a generalisation of maxitivity and minitivity properties of possibility theory, thus offering a classification of qualitative capacities in terms of increasing levels of complexity and generality, based on the minimal number of possibility distributions needed to represent them. In particular, it has been shown that a Sugeno integral is a lower possibility integral [7]. Then the computation of Sugeno integral can be reduced for k-adjunctive or k-max dominant capacities. Moreover, the study of relationships between the focal sets of a capacity and the focal sets of its conjugate has shown the links between k-adjunction and k-maxitive capacities. We have finally shown a connection between qualitative capacities and non-regular modal logics, which generalize KD-style modal logics in the same sense as capacities generalize necessity measures.

Numerous alleys of research are opened by the above results:

- On the logical side, we may reconsider the study of non-regular modal logics in the light of capacity-based semantics. The fact that they lead to disjunctions of KD necessity operators is clearly reminding of Belnap epistemic set-up [3], and paraconsistent logics. The fact that an extreme case of the EMN logic comes down to a modal logic similar to a KD one where possibility and necessity are exchanged reflects the fact that in Belnap bilattices, the epistemic values representing conficting information and absence thereof play symmetric roles
- One may also wish to evaluate the quantity of information (or uncertainty) contained in a qualitative capacity [16]. In [7], the maximal specific possibility distribution dominating a capacity was studied and shown to be the counterpart of the contour function of belief functions for qualitative capacities. This notion could suggest one approach based on the comparison of contour functions.
- The analogy between belief functions and qualitative capacities was discussed in [18] and a qualitative counterpart of information ordering based on specialisation (inclusion of focal sets) was also proposed, as well as counterparts to Dempster rule of combination. These lines should be pursued in the scope of qualitative information fusion techniques going beyond those based on possibility theory.

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